Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	92	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (92 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

(a) Determine the vector projection of $\begin{pmatrix}3\\4\end{pmatrix}$ on $\begin{pmatrix}-2\\1\end{pmatrix}$. (2 marks)

Solution
$\dfrac{\begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}}{\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}}\begin{pmatrix}-2\\1\end{pmatrix} = \dfrac{-2}{5}\begin{pmatrix}-2\\1\end{pmatrix}$ $= \begin{pmatrix}4/5\\-2/5\end{pmatrix}$ $= \begin{pmatrix}0.8\\-0.4\end{pmatrix}$
Specific behaviours
✓ indicates appropriate method ✓ correct vector

(b) Determine the exact value(s) of t so that the vectors $\begin{pmatrix}t\\-1\end{pmatrix}$ and $\begin{pmatrix}3t-5\\2\end{pmatrix}$ are

(i) parallel. (2 marks)

Solution
$\dfrac{t}{-1} = \dfrac{3t-5}{2} \Rightarrow t = \dfrac{10}{7}$
Specific behaviours
✓ indicates equation using of ratio of coefficients ✓ correct value

(ii) perpendicular. (2 marks)

Solution
$\begin{pmatrix}t\\-2\end{pmatrix}\cdot\begin{pmatrix}3t-5\\1\end{pmatrix} = 0$ $3t^2 - 5t - 2 = 0$ $t = 2, \quad t = -\dfrac{1}{3}.$
Specific behaviours
✓ indicates equation using scalar product ✓ correct values

Question 10 **(7 marks)**

Two transformation matrices are $\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\boldsymbol{S} = \begin{bmatrix} 2 & 5 \\ -2 & 5 \end{bmatrix}$.

Triangle ABC has an area of 35 cm^2, with vertices at $A(-3, -5), B(6, -6)$ and $C(4, 2)$.

(a) Determine the coordinates of ABC after the triangle has been transformed by matrix $\boldsymbol{T}$. (3 marks)

Solution
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -3 & 6 & 4 \\ -5 & -6 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 4 \\ 5 & 6 & -2 \end{bmatrix}$ $A'(-3, 5), \quad B'(6, 6), \quad C'(4, -2).$
Specific behaviours
✓ indicates pre-multiplication by $\boldsymbol{T}$ ✓ correct matrix product   ✓ correctly lists set of coordinates

(b) Use the geometric transformation to explain why the determinant of $\boldsymbol{T}$ is 1. (1 mark)

Solution
$\boldsymbol{T}$ represents a reflection, the area of triangle does not change and so determinant is 1.
Specific behaviours
✓ reflection will not change area

(c) Use the geometric transformation to explain why $\boldsymbol{T}^2 = \boldsymbol{I}$, where $\boldsymbol{I}$ is the 2×2 identity matrix. (1 mark)

Solution
$\boldsymbol{T}^2$ represents two reflections in the same line, and so the triangle will be back where it started, with the same coordinates.
Specific behaviours
✓ two reflections in same line

(d) Determine the area of ABC after the triangle has been transformed by matrix $\boldsymbol{S}$. (2 marks)

Solution
$\det(\boldsymbol{S}) = 10 + 10 = 20$ $\text{New area} = 20 \times 35 = 700 \text{ cm}^2$
Specific behaviours
✓ calculates determinant   ✓ calculates new area

Question 11 **(7 marks)**

(a) Five-digit even numbers are to be made using the digits $2, 3, 4, 5, 6$ and 7. Determine how many such numbers exist if the number must exceed $40\,000$ and no digit may be used more than once in a number. (3 marks)

Solution
End with 2: $1 \times 4 \times 4 \times 3 \times 2 = 96$ End with 4 or 6: $2 \times 3 \times 4 \times 3 \times 2 = 144$ Total possible numbers: $144 + 96 = 240$
Specific behaviours
✓ splits into mutually exclusive cases ✓ correctly counts at least one case ✓ calculates total

(b) The library in a small guesthouse has 28 different books, of which 19 are fiction and the remainder non-fiction. Determine the number of different ways that a guest can select four books if they want

(i) the same number of fiction and non-fiction books. (2 marks)

Solution
$n = \binom{19}{2}\binom{9}{2}$ $= 171 \times 36$ $= 6156 \text{ ways}$
Specific behaviours
✓ indicates correct method ✓ correct number of ways

(ii) at least one non-fiction book. (2 marks)

Solution
No non-fiction: $n = \binom{19}{4}\binom{9}{0} = 3876$ No restrictions: $n = \binom{28}{4} = 20\,475$ At least one non-fiction: $20\,475 - 3876 = 20\,349$ ways.
Specific behaviours
✓ indicates correct method ✓ correct number of ways

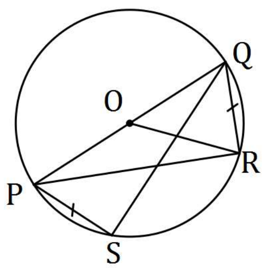
Question 12 **(8 marks)**

(a) Write the inverse of the true statement 'if a figure is a square then it is a quadrilateral' and use an example or counter-example to briefly discuss the truth of the inverse.

(2 marks)

Solution
Inverse: If a figure is not a square, then it is not a quadrilateral. The inverse is false, as the figure could be a rectangle.
Specific behaviours
✓ correct inverse ✓ states false with counter-example

(b) Points P, Q, R and S lie as shown on a circle with centre O so that PQ is a diameter, $PS = QR$ and $\angle QOR = 36°$.



Determine the size of

(i) $\angle QPR$. (1 mark)

Solution
$\angle QPR = 36° \div 2 = 18°$ $\angle QPS = \angle RQP = 90° - 18° = 72°$ $\angle RPS = 72° - 18° = 48°$
Specific behaviours
✓ $\angle QPR$ ✓ $\angle QPS$ ✓ $\angle RPS$

(ii) $\angle QPS$. (1 mark)

(iii) $\angle RPS$. (1 mark)

(c) Two chords of a circle, AC and BD, intersect at E so that $AC = 59$ cm, $BE = 33$ cm and $BD = 53$ cm. Determine all possible lengths of AE. (3 marks)

Solution
Using intersecting chord theorem, $AE \times EC = BE \times ED$. $ED = BD - BE = 53 - 33 = 20$ Let $AE = x$, so that $CE = 59 - x$ $33 \times 20 = x(59 - x)$ $x = 15, 44$ Hence $AE = 15$ cm or $AE = 44$ cm.
Specific behaviours
✓ identifies required lengths ✓ forms quadratic equation ✓ states both values

Question 13 **(8 marks)**

In triangle OAB, P is the midpoint of OB and M is the midpoint of PA. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Show that $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{4}\mathbf{b}$. (2 marks)

Solution
$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}, \quad \overrightarrow{OP} = \frac{1}{2}\overrightarrow{OB}, \quad \overrightarrow{PM} = \frac{1}{2}\overrightarrow{PA} = \frac{1}{2}\left(\overrightarrow{OA} - \overrightarrow{OP}\right)$ $= \frac{1}{2}\mathbf{b} + \frac{1}{2}\left(\mathbf{a} - \frac{1}{2}\mathbf{b}\right)$ $= \frac{1}{2}\mathbf{a} + \frac{1}{4}\mathbf{b}$
Specific behaviours
✓ indicates logical steps   ✓ uses correct vector notation throughout

The position vector of A is $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$, position vector of B is $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ and O is the origin.

(b) Determine a unit vector $\hat{\mathbf{u}}$ in the same direction as $\overrightarrow{OM}$. (2 marks)

Solution
$\overrightarrow{OM} = \frac{1}{2}\begin{pmatrix} 5 \\ -5 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $\hat{\mathbf{u}} = \frac{\sqrt{5}}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **NB might use CAS for last step*
Specific behaviours
✓ calculates $\overrightarrow{OM}$ ✓ states unit vector

(c) Show that OB is perpendicular to PM. (2 marks)

Solution
$\overrightarrow{OB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \quad \overrightarrow{PM} = \frac{1}{2}\left(\begin{pmatrix} 5 \\ -5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\overrightarrow{OB} \cdot \overrightarrow{PM} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 6 - 6 = 0$ Hence vectors are perpendicular.
Specific behaviours
✓ calculates $\overrightarrow{PM}$ ✓ shows scalar product is zero

(d) Determine the size of $\angle AOB$. (2 marks)

Solution
$\angle AOB = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}\right)$ $= \cos^{-1}\left(\frac{\sqrt{5}}{5}\right)$ $= 63.4°$ **NB might use CAS*
Specific behaviours
✓ indicates suitable method   ✓ correct angle

Question 14 **(6 marks)**

Consider the following statement:

For two integers a, b if $5a^2 + 2b^2$ is a multiple of 4 then at least one of a, b is even.

(a) Write the contrapositive of the statement. (1 mark)

Solution
For two integers a, b if both are odd then $5a^2 + 2b^2$ is not a multiple of 4.
Specific behaviours
✓ correct contrapositive

(b) Prove that the statement is true. (5 marks)

Solution

Proof of contrapositive:

If a, b both odd, then $a = 2m + 1$ and $b = 2n + 1$, where n, m both integers.

Hence

$$\begin{aligned} 5a^2 + 2b^2 &= 5(2m+1)^2 + 2(2n+1)^2 \\ &= 20m^2 + 20m + 8n^2 + 8n + 7 \\ &= 4(5m^2 + 5m + 2n^2 + 2n + 2) - 1 \end{aligned}$$

Hence $5a^2 + 2b^2$ will never be a multiple of 4 as it is always one less than a multiple of 4.

Since the contrapositive statement has been proved to be true then it follows that the original statement must also be true.

Specific behaviours

✓ attempts to prove contrapositive and states truth of contrapositive implies truth of original statement
✓ uses form $2k + 1$ form for odd numbers a, b
✓ substitutes for a, b and expands
✓ factors out 4
✓ explains why contrapositive true

Question 15 **(8 marks)**

The height of the tide, h cm, of the sea above the mean level at time t hours after midnight one day is given by

$$h(t) = 165\sin\left(\frac{\pi t}{6}\right) - 52\cos\left(\frac{\pi t}{6}\right).$$

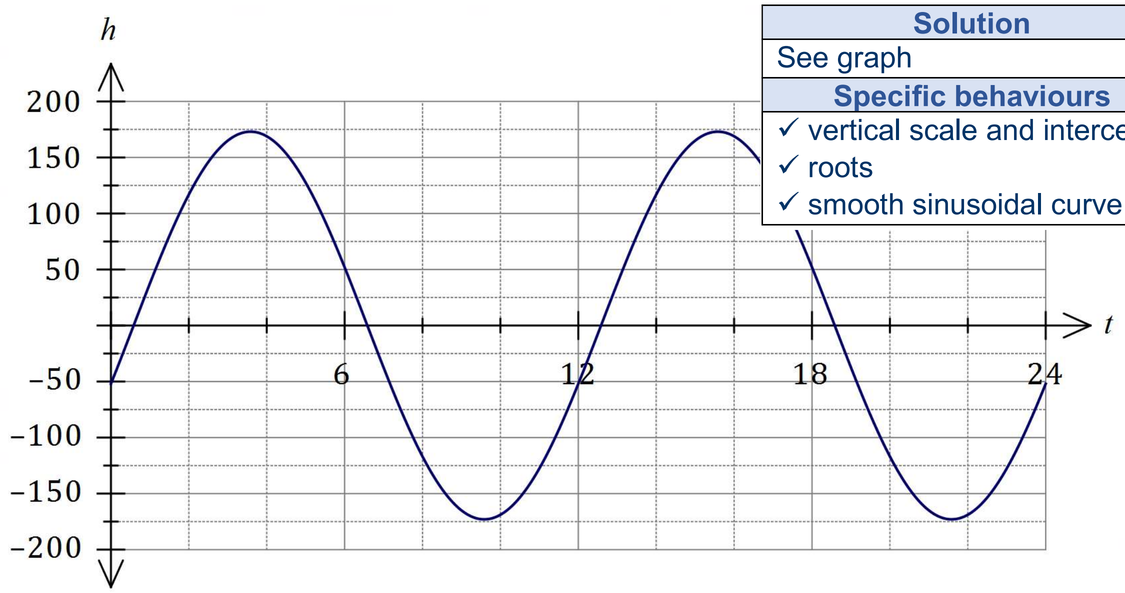
(a) Express h in the form $a\sin(bt - \theta)$, where $a, b > 0$ and $0 \le \theta \le 2\pi$. (3 marks)

Solution
$a = \sqrt{165^2 + 52^2} = 173$ $h = 173\sin\left(\frac{\pi t}{6} - \theta\right)$ $= 173\left(\sin\left(\frac{\pi t}{6}\right)\cos\theta - \cos\left(\frac{\pi t}{6}\right)\sin\theta\right)$ $\cos\theta = \frac{165}{173}, \sin\theta = \frac{52}{173} \Rightarrow \theta = 0.3053$ $h = 173\sin\left(\frac{\pi t}{6} - 0.3053\right)$
Specific behaviours
✓ value of a ✓ value of θ ✓ correct expression for h

(b) Determine, to the nearest minute, the time of the first high tide. (2 marks)

Solution
Require $\frac{\pi t}{6} - 0.3053 = \frac{\pi}{2} \Rightarrow t = 3.583$ h. Hence at 0335 or 3:35 am.
Specific behaviours
✓ time in hours   ✓ time of day, to nearest minute

(c) Sketch the graph of the height of the tide on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ vertical scale and intercept   ✓ roots   ✓ smooth sinusoidal curve

Question 16 **(7 marks)**

(a) In trapezium $OABC$, AC and OB are diagonals, and AB is parallel to OC. Use a vector method to prove that $\overrightarrow{AC} + \overrightarrow{OB} = \overrightarrow{AB} + \overrightarrow{OC}$. (3 marks)

Solution
Note that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{OB} = \overrightarrow{OC} - \overrightarrow{BC}$. $$\begin{aligned} LHS &= \overrightarrow{AC} + \overrightarrow{OB} \\ &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{OC} - \overrightarrow{BC} \\ &= \overrightarrow{AB} + \overrightarrow{OC} \\ &= RHS \end{aligned}$$
Specific behaviours
✓ expression for one diagonal ✓ expression for other diagonal, using an opposite ✓ adds expressions and simplifies

(b) In rectangle $PQRS$, let $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$. Use a vector method to prove that if the diagonals PR and QS are perpendicular, then $PQRS$ is a square. (4 marks)

Solution
The diagonals are the vectors $\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{QS} = \mathbf{b} - \mathbf{a}$. If diagonals are perpendicular, then $\overrightarrow{PR} \cdot \overrightarrow{QS} = 0$ and so: $$\begin{aligned} (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) &= 0 \\ \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} &= 0 \\ \lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2 &= 0 \\ \lvert\mathbf{b}\rvert^2 = \lvert\mathbf{a}\rvert^2 &\Rightarrow \lvert\mathbf{b}\rvert = \lvert\mathbf{a}\rvert \end{aligned}$$ Hence $PQRS$ must be a square since it is a rectangle with equal length sides.
Specific behaviours
✓ determines vectors for diagonals ✓ uses scalar product ✓ expands and simplifies scalar product ✓ explains that sides are equal length

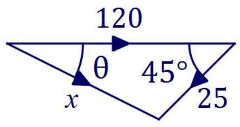
Question 17 **(7 marks)**

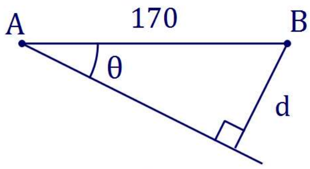
Airport B lies 170 km due east of airport A, and in the region of the airports a wind of 25 km/h is blowing from the northeast.

A small plane, with a cruising speed of 120 km/h, leaves airport A. The pilot, not aware of the wind and intending to fly to airport B, steered the plane on a bearing of 090°.

Assuming that the pilot does not realise their mistake, determine how close the plane will come to airport B if it continues to fly for several hours on the same bearing.

Solution





$$x^2 = 120^2 + 25^2 - 2(120)(25)\cos 45°$$
$$x = 103.84$$

$$\frac{\sin\theta}{25} = \frac{\sin 45°}{103.84}$$
$$\theta = 9.80°$$

$$d = 170\sin 9.8°$$
$$= 28.9 \text{ km}$$

Specific behaviours

- ✓ appropriate sketch/explanation for x and θ
- ✓ equation for x
- ✓ solves for x
- ✓ equation for θ
- ✓ solves for θ
- ✓ appropriate sketch/explanation for closest distance
- ✓ calculates closest distance

Question 18 **(7 marks)**

Three forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ act on a small body, where $\mathbf{F}_1 = 4\mathbf{i} - 10\mathbf{j}$ N, $\mathbf{F}_2 = -8\mathbf{i} + 16\mathbf{j}$ N and $\mathbf{F}_3 = 9\mathbf{i} - 15\mathbf{j}$ N.

(a) Determine the magnitude of the resultant force and the angle between the resultant force and the vector $\mathbf{i}$. (3 marks)

Solution
$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$
$= \begin{pmatrix} 4 \\ -10 \end{pmatrix} + \begin{pmatrix} -8 \\ 16 \end{pmatrix} + \begin{pmatrix} 9 \\ -15 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \end{pmatrix}$
$\lvert\mathbf{R}\rvert = \sqrt{5^2 + 9^2} = \sqrt{106} \approx 10.3$ N
$\angle = \tan^{-1}\left(\frac{-9}{5}\right) \approx -60.9°$
Hence resultant has a magnitude of 10.3 N and makes an angle of 60.9° with $\mathbf{i}$.
Specific behaviours
✓ correct sum in component form ✓ calculates magnitude ✓ calculates angle

(b) Two of the forces, $\mathbf{F}_2$ and $\mathbf{F}_3$, can be multiplied by scalars λ and μ respectively so that the three forces are in equilibrium. Determine the value of λ and the value of μ. (4 marks)

Solution
$\mathbf{F}_1 + \lambda\mathbf{F}_2 + \mu\mathbf{F}_3 = \mathbf{0}$
$\begin{pmatrix} 4 \\ -10 \end{pmatrix} + \lambda\begin{pmatrix} -8 \\ 16 \end{pmatrix} + \mu\begin{pmatrix} 9 \\ -15 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Resolving in i and j directions:
$4 - 8\lambda + 9\mu = 0$
$-10 + 16\lambda - 15\mu = 0$
Solving simultaneously gives
$\lambda = \frac{5}{4} = 1.25, \quad \mu = \frac{2}{3}$
Specific behaviours
✓ writes vector equation equal to 0 ✓ forms two equations ✓ value of λ ✓ value of μ

Question 19 **(7 marks)**

(a) 222 people are asked to choose two different letters from those in the word PROBATIVE and write them down in order. Use the pigeonhole principle to prove that at least four people will write the same pair of letters in the same order. (3 marks)

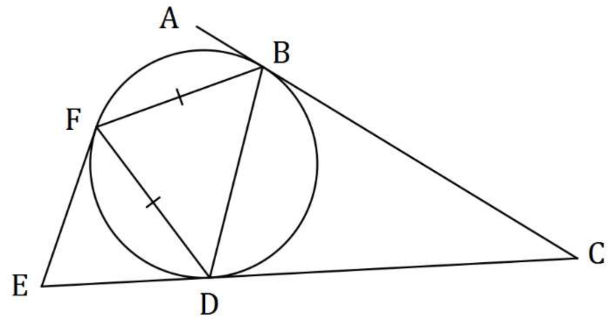
Solution
There are ${}^9P_2 = 72$ different ordered pairs of letters, each of which is a pigeonhole. Using the pigeonhole principle, if 222 pigeons (the number of pairs of letters written by people) are placed into 72 pigeonholes, then at least one pigeonhole will contain $\lceil 222 \div 72 \rceil = 4$ or more pigeons. Hence at least 4 people will write the same pair of letters in the same order.
Specific behaviours
✓ obtains number of permutations ✓ indicates pigeons and pigeonholes ✓ uses pigeonhole principle to complete proof

(b) Three character codes, such as PHN, are made using three different letters chosen from the word CHAMPIONED. Determine the proportion of all possible codes that start with a C or end with a D. (4 marks)

Solution
Start with C: $\text{n}(C) = 1 \times {}^9P_2 = 72$ codes. End with D: $\text{n}(D) = 1 \times {}^9P_2 = 72$ codes. Start with C and end with D: $\text{n}(C \cap D) = 1 \times 1 \times {}^8P_1 = 8$ codes. Start with C or end with D: $\text{n}(C \cup D) = 72 + 72 - 8 = 136$ codes. There are ${}^{10}P_3 = 720$ different codes. Hence required proportion is $\dfrac{136}{720} = \dfrac{17}{90} \approx 0.189.$
Specific behaviours
✓ $\text{n}(C)$ and $\text{n}(D)$ ✓ $\text{n}(C \cap D)$ ✓ $\text{n}(C \cup D)$ ✓ number of possible codes and writes proportion

Question 20 **(7 marks)**

The diagram, not drawn to scale, shows vertices B, D and F of an isosceles triangle lying on a circle so that $FD = FB$.



Lines CA and CE are tangential to the circle at B and D respectively.

$CBFE$ is a cyclic quadrilateral.

Let $\angle ACE = \alpha$.

(a) Determine, with reasons, the size of $\angle FEC$ in terms of α. (5 marks)

Solution
$\angle CBD = \frac{1}{2}(180° - \alpha) = 90° - \frac{1}{2}\alpha$ (tangents from $C \Rightarrow CB = CD$) $\angle BFD = \angle CBD = 90° - \frac{1}{2}\alpha$ (alternate segment) $\angle FBD = \frac{1}{2}\left(180° - \left(90° - \frac{1}{2}\alpha\right)\right) = 45° + \frac{1}{4}\alpha$ (isosceles triangle) $\angle CBF = \angle CBD + \angle FBD$ (adjacent angles) $= 90° - \frac{1}{2}\alpha + 45° + \frac{1}{4}\alpha = 135° - \frac{1}{4}\alpha$ $\angle FEC = 180° - \angle CBF$ (cyclic quadrilateral) $= 180° - \left(135° - \frac{1}{4}\alpha\right) = 45° + \frac{1}{4}\alpha$
Specific behaviours
✓ expression for $\angle CBD$ with reason   ✓ expression for $\angle BFD$ with reason   ✓ expression for $\angle FBD$ with reason   ✓ expression for $\angle CBF$ with reason   ✓ expression for $\angle FEC$ with reason

(b) Hence determine the range of values for the size of $\angle FEC$ in degrees. (2 marks)

Solution
For figure to exist, $0° < \alpha < 180°$.   Hence $\angle FEC > 45° + \frac{1}{4}(0°)$ and $\angle FEC < 45° + \frac{1}{4}(180°)$ Range is $45° < \angle FEC < 90°$.
Specific behaviours
✓ indicates correct domain for α ✓ correct range, including inequalities

Question 21 **(7 marks)**

Triangle XYZ has vertices $X(6,-2), Y(-5,5)$ and $Z(2,9)$.

XYZ is reflected in the line $y=-\dfrac{x}{2}$ to form triangle $X'Y'Z'$.

(a) Determine the coordinates of Y'. (2 marks)

Solution
$$\frac{1}{5}\begin{bmatrix}3 & -4\\ -4 & -3\end{bmatrix}\begin{bmatrix}-5\\ 5\end{bmatrix}=\begin{bmatrix}-7\\ 1\end{bmatrix}$$ $$Y'(-7,1)$$
Specific behaviours
✓ reflection matrix ✓ correct coordinates

XYZ is rotated clockwise $45°$ about the origin to form triangle $X''Y''Z''$.

(b) Determine the coordinates of X''. (2 marks)

Solution
$$\frac{1}{2}\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ -\sqrt{2} & \sqrt{2}\end{bmatrix}\begin{bmatrix}6\\ -2\end{bmatrix}=\begin{bmatrix}2\sqrt{2}\\ -4\sqrt{2}\end{bmatrix}\approx\begin{bmatrix}2.828\\ -5.657\end{bmatrix}$$ $$X''\left(2\sqrt{2},-4\sqrt{2}\right)$$
Specific behaviours
✓ rotation matrix ✓ correct coordinates

(c) Determine matrix $\boldsymbol{M}$ that will transform $X''Y''Z''$ to $X'Y'Z'$. (3 marks)

Solution
Matrix $\boldsymbol{P}$ for $X''Y''Z''\rightarrow XYZ$ is inverse of that used in (b). Matrix $\boldsymbol{Q}$ for $XYZ\rightarrow X'Y'Z'$ is same as used in (a). Hence $$\boldsymbol{M}=\boldsymbol{QP}^{-1}$$ $$=\frac{1}{5}\begin{bmatrix}3 & -4\\ -4 & -3\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ -\sqrt{2} & \sqrt{2}\end{bmatrix}^{-1}$$ $$=\frac{1}{10}\begin{bmatrix}-\sqrt{2} & -7\sqrt{2}\\ -7\sqrt{2} & \sqrt{2}\end{bmatrix}\approx\begin{bmatrix}-0.141 & -0.990\\ -0.990 & 0.141\end{bmatrix}$$
Specific behaviours
✓ matrix for $X''Y''Z''\rightarrow XYZ$ ✓ indicates correct order of multiplication ✓ correct matrix $\boldsymbol{M}$

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________